

Detection efficiency loophole and Pusey-Barrett-Rudolph theorem

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Detection efficiency loophole poses a significant problem for experimental tests of Bell inequalities. Recently discovered Pusey-Barrett-Rudolph (PBR) theorem suffers from the same vulnerability. In this paper we calculate the critical detection efficiency, below which the PBR argument for the ontic nature of quantum state is inconclusive. This is done for the maximally ψ -epistemic models. We use two different definitions of this property. The optimal number of parties, for which the critical detection efficiency is the lowest is given. We also approach the problem from the opposite direction. We provide a function which enables us to specify which epistemic models are ruled out by the results of an experiment with a given detection efficiency.

I. INTRODUCTION

The status of the quantum state is a topic of a long standing discussion. There are many views on this issue in physics community. The ensemble one: the quantum state is a theoretical description of a statistical ensemble of equivalently prepared systems and there are no underlying states for individual systems (see e.g. [1]). Another school of thought, the *ontic* interpretation, sees quantum state as a real physical object, an inherent property of an individual quantum system. Underlying states exist, but they uniquely point to the quantum state. A competing point of view, the *epistemic* one, considers, the quantum state as a mere mathematical object to calculate probability of certain events. It is a state of knowledge, [2]. However, a more basic description of the system is possible involving real physical states of individual systems, or hidden variables. The principal technical difference between the ontic and epistemic approach, is that in the later case two different, but non-orthogonal, quantum states may be linked with the same “physical state”. Bell’s Theorem tells us that a description with such variables cannot be local, however it does not exclude the possibility of non-local theories.

The Pusey-Barrett-Rudolph (PBR) Theorem [3], is a major advance in the studies of the relation between ontic and epistemic views. It states that, one can find experimental situations, for which quantum mechanical predictions force us, if we allow hidden variables or states, to adopt the ontic interpretation of the quantum state. This result got much attention and experiments followed [4–6]. However, like all experiments probing foundations of quantum mechanics the PBR proposal suffers from the detection efficiency loophole [7], i.e. the possibility that in the actual experiment the seemingly quantum effect is due to a post-selection of the events. Only the subsample of the cases in which all detectors stations register particles may be following the quantum predictions. In fact the effect of detection efficiency loophole is much more severe in the case of PBR than for Bell inequalities. In the latter case, for any “relevant” inequality there exists a critical detection efficiency, which, if attained in the experiment, rules out the local hidden variable de-

scription. In the former however the efficiency must be 100%. This makes PBR theorem not viable for experimental tests, see footnote [22], unless some additional assumptions are made about hidden variables. Such assumptions lead to finite critical efficiencies. The purpose of this paper is to study the critical minimal detection efficiency, required for the PBR experiment to be conclusive, under certain reasonable conditions imposed on the distributions of hidden variables.

First we present a short description of PBR argument. Next we discuss detection efficiency loophole in the context of Bell inequalities and explain why for PBR its effect is so much stronger. Finally we move to the main part of our paper. We find the critical detection efficiency for the following “reasonable” additional assumption: hidden variable distributions associated with two different quantum states are required to have the same overlap as the states. Even with this assumption we obtain a very high critical detection efficiency, below which experimental tests, based on PBR argument on ontic nature of quantum state, are inconclusive.

II. PBR THEOREM

Let us give a short presentation of the results of Pusey et al. [3]. Within the *non-orthodox view* allowing for hidden states or variables, the authors have shown a unique relationship between hidden-physical reality and the quantum state.

Let’s consider two different preparation procedures resulting in two quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$. The states are assumed to be different, but do not have to be orthogonal. The underlying hidden-physical states will be denoted by λ . One can assume that there exists a probability distribution $\rho_\psi(\lambda)$ in some ontic space R associated with the given state $|\psi\rangle$. If supports of $\rho_{\psi_1}(\lambda)$ and $\rho_{\psi_2}(\lambda)$ overlap, then there is at least one hidden-physical state common to both distributions. However, if the supports do not overlap, then they do not share any common hidden-physical state. Pusey et al. in their argument have put forward a gedanken-experiment, for which, if underlying hidden variable model can explain the probabilistic nature of quantum mechanics, $\rho_{\psi_1}(\lambda)$ and $\rho_{\psi_2}(\lambda)$

must have disjoint supports. This implies to the ontic nature of the state $|\psi\rangle$, as hidden variables pinpoint the state with which they are associated.

In general, if measurement outcome depends on hidden-physical state λ , then for a given state $|\psi_1\rangle$ the probability to obtain an outcome $|\psi_2\rangle$ for a measurement $M = |\psi_2\rangle\langle\psi_2|$, associated with a response function $\xi_M(\psi_2|\lambda)$ reads

$$P(\psi_2|\psi_1) = \int_R \xi_M(\psi_2|\lambda) \rho_{\psi_1}(\lambda) d\lambda, \quad (1)$$

where $0 \leq \xi_M(\psi_2|\lambda) \leq 1$. If hidden variables can reproduce quantum mechanical predictions, then by Born rule, $P(\psi_2|\psi_1) = |\langle\psi_2|\psi_1\rangle|^2$. The authors have considered a specific joint measurement on a composite system with n independently prepared sub-systems. They show that, if probability distributions of hidden-physical states corresponding to two different quantum states overlap, the common hidden-physical states from the overlap region must lead to measurement outcomes which are forbidden by quantum mechanical predictions. A diagram of a quantum circuit used in the gedanken-experiment [3] is presented in Fig. 1.

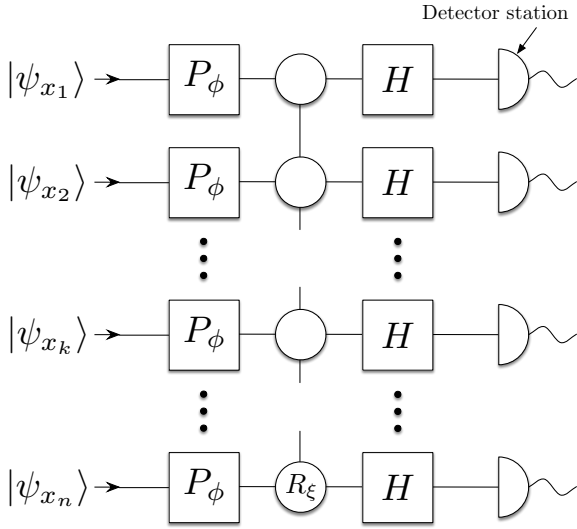


FIG. 1: PBR-argument is based on a joint measurement in a quantum circuit of n -qubits followed by a measurement on each qubit in the computational basis. The single qubit gates are defined as $P_\phi = |0\rangle\langle 0| + e^{i\phi}|1\rangle\langle 1|$ and the Hadamard gate, $|+\rangle\langle 0| + |-\rangle\langle 1|$. The entangling gate in the middle rotates only $|00\dots 00\rangle$, $R_\xi|00\dots 00\rangle = e^{i\xi}|00\dots 00\rangle$. Each of the outputs is observed by a single detector station.

The construction of [3] runs as follows. Each sub-system is prepared under conditions which exclude any interdependence between the sub-systems (e.g., the are prepared in mutually remote locations, at the same moment of time, etc.). The following 2^n possible states are considered :

$$|\psi_{x_1, x_2, \dots, x_n}\rangle = |\psi_{x_1}\rangle \otimes |\psi_{x_2}\rangle \otimes \dots \otimes |\psi_{x_n}\rangle, \quad (2)$$

where $x_i \in 1, 2$ and $|\psi_1\rangle = \cos \frac{\theta}{2}|0\rangle + \sin \frac{\theta}{2}|1\rangle$ and $|\psi_2\rangle = \cos \frac{\theta}{2}|0\rangle - \sin \frac{\theta}{2}|1\rangle$.

If there exists a hidden-physical state λ , for which $\rho_{\psi_1}(\lambda) > 0$ and $\rho_{\psi_2}(\lambda) > 0$ then there is a non-zero probability that $\psi_{x_1, x_2, \dots, x_n}$ will be prepared in the state $\lambda^{\otimes n}$ (this notation means: all systems in the hidden state λ) regardless of the choices of x_i 's. The measurement is chosen in such a way that, according to quantum predictions, for every choice of x_1, x_2, \dots, x_n , one outcome, different for each $\psi_{x_1, x_2, \dots, x_n}$, is prohibited (i.e. its probability is zero). The number of possible outcomes is the same as the number of possible preparations. Thus, if the hidden variable theory is to be in agreement with quantum mechanics then $\lambda^{\otimes n}$ must lead to probability zero for any outcome! Therefore, $\lambda^{\otimes n}$ can not be the underlying state linked to any $\psi_{x_1, x_2, \dots, x_n}$. We have a contradiction. This implies that the premise, i.e existence of λ , for which $\rho_{\psi_1}(\lambda) > 0$ and $\rho_{\psi_2}(\lambda) > 0$ is wrong, which leads us to the conclusion that there is unique correspondence between any hidden state λ and the quantum state with which it is associated. That is the quantum state is of an *ontic* nature.

However, this argument explicitly assumes that the particles are always detected. If there exists a mechanism, which makes at least one of the detectors not click when the measurement is done on the hidden state $\lambda^{\otimes n}$, then there is no contradiction. This is the detection efficiency loophole. Since PBR argument does not provide us with any method of estimating the probability with which $\lambda^{\otimes n}$ is generated, we have no way of estimating how rarely the detectors should fail to click to avoid this loophole. Therefore, extra assumptions are necessary.

III. BELL'S INEQUALITIES AND DETECTION LOOPHOLE

To experimentally refute the possibility of local, realistic description of quantum systems one has to violate a Bell inequality making sure that the conditions used to derive them are satisfied. Such conditions include the specific properties of the considered Bell experiment. If the experiment has specific features which are different than the ones assumed in derivation of Bell's inequalities, which lead to inconclusiveness of the experiment, then we talk about loopholes. One of them is the detection efficiency loophole [8].

In [9] an explicit local model is given which mimics quantum correlations for a singlet state and projective measurements, provided the detection efficiency is below 67%. For every Bell inequality there exists a threshold minimal detection efficiency, which is required, if one wants to reject local-realism by violating the inequality. For the simplest case of two binary measurements on two-qubit entangled state the best known result is due to Eberhard [10] who showed that the critical detection efficiency for CH inequality[11] is $\frac{2}{3}$. For systems of higher dimension the threshold decreases exponentially with the

dimension [12]. For multipartite Bell inequalities Cabello et al. [13] have shown that a detection efficiency of $\frac{n}{2n-1}$ is both necessary and sufficient to violate n -partite Mermin [14] inequalities.

In the next section we discuss similar problems for the PBR gedanken-experiment.

IV. DETECTION EFFICIENCY LOOPHOLE IN PBR THEOREM

Our aim is to find the critical detection efficiency below which PBR's argument on ontic nature of quantum states does not work anymore.

If ontic description is to be true, then each hidden-physical state must be uniquely linked to a *single* specific quantum state. If two different probability distributions of hidden-physical states corresponding to two different quantum states partially overlap, then there is an ambiguity in one to one relationship between hidden-physical state and quantum state. More precisely, there is a non zero probability that two different preparation methods corresponding to two quantum states may lead to same hidden-physical state. If there are n subsystems and one considers the 2^n possible combinations of states (2), for the epistemic approach there are an underlying hidden-physical states of the joint system $\lambda_0^{\oplus n}$, which can correspond to any of these. To save ψ -epistemic models from contradicting quantum predictions it suffices that one of the detectors used in the experiment fails to click whenever the measured compound system is in $\lambda_0^{\oplus n}$.

Before we start our analysis we state that, we assume, that the detection inefficiency is the only experimental imperfection that we take into account.

Let p be the probability that the supports of two probability distributions $\rho_{\psi_1}(\lambda)$ and $\rho_{\psi_2}(\lambda)$ overlap. In the next section, we estimate p as a measure of *epistemic overlap* (see Ref [16]) between two quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$. For simplicity we assume that the value of p does not depend on the n -system quantum state $\psi_{x_1, x_2, \dots, x_n}$, which is prepared. Thus the total probability, associated with the overlap of two distributions $\rho_{\psi_1}(\lambda)$, and $\rho_{\psi_2}(\lambda)$ for n independently prepared subsystems in state $\psi_{x_1, x_2, \dots, x_n}$, is $p_1 = p^n$.

In the circuit shown in Fig. 1 there are as many detector stations as subsystems. Assume that the detectors have detection efficiency η and their detection probabilities are independent. Then, the probability that at least one of the detection stations registers no click is $1 - \eta^n$. If the detection loophole is to be solely responsible for the fact that no outcomes which contradict quantum mechanics are registered, then p_1 has to be lower or equal $1 - \eta^n$. Thus, the critical detection efficiency is given by

$$\eta = (1 - p^n)^{\frac{1}{n}}. \quad (3)$$

One can find in Ref.[3] that values of the angle θ , which defines $|\psi_1\rangle$ and $|\psi_2\rangle$, and their scalar product $\langle\psi_1|\psi_2\rangle = \cos\theta$, determine how many qubits one has to have, so that

the PBR argument for gedanken-experiment of Fig. 1 can work. If one denotes the the number of qubits by n , then the relation is given by

$$0 < 2 \arctan(2^{\frac{1}{n}} - 1) \leq \theta < \frac{\pi}{2}. \quad (4)$$

Thus we have a functional relation between the minimal value of θ in the gedanken-experiment, and the number of qubits:

$$n(\theta_{\min}) = \left\lceil \frac{1}{\log_2(1 + \tan(\frac{\theta_{\min}}{2}))} \right\rceil, \quad (5)$$

Therefore, if one, under some assumptions, can fix p (see e. g. next section) by combining (3) and (5) one can get the minimal efficiency η as a function of θ .

Because PBR theorem does not say anything about how big p can be, it can be in principle arbitrarily small. Thus, the critical efficiency given by (3) can be arbitrarily close to 1. However this will not be the case if one assumes additionally some specific relation between p and $\langle\psi_2|\psi_1\rangle$.

V. MAXIMALLY ψ -EPISTEMIC MODELS

The additional assumption that we now make is that the model that we are trying to falsify is maximally ψ -epistemic [15]. By definition of Ref. [15], it implies that, see Fig. 2

$$\begin{aligned} p &= \int_{R_{\psi_1 \psi_2}} \rho_{\psi_1}(\lambda) d\lambda \\ &= \int_{R_{\psi_1 \psi_2}} \rho_{\psi_2}(\lambda) d\lambda = |\langle\psi_2|\psi_1\rangle|^2 = \cos^2 \theta. \end{aligned} \quad (6)$$

For the maximally ψ -epistemic models the union of the supports of any quantum basis states spans the whole space of hidden-physical states. Note that this implies that in the case of a two dimensional subspace, any pair of orthogonal states has the same union of supports. Thus we necessarily have non-zero overlaps for ρ_{ψ_1} and ρ_{ψ_2} , see Fig. 2, provided are linked with two non-orthogonal states.

A. Alternative approaches

The validity of the PBR argument rests crucially on *preparation independence* assumption, which means that the probability distributions of λ 's corresponding to two different systems are independent. Recently there have been several attempts to re-derive PBR result without this assumption but with limited success [16–19]. While PBR argument works for any pair of states, results of [16–19] are not so general. The states for which these arguments hold are of a certain dimension (at least 3 in all

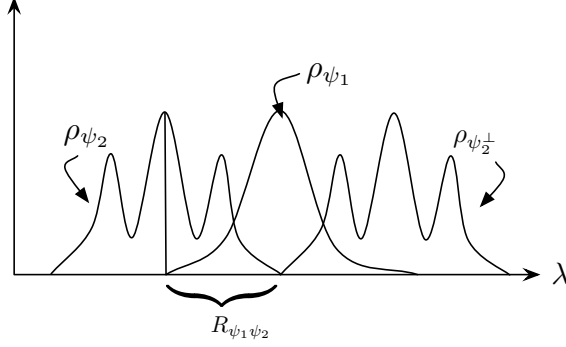


FIG. 2: A schematic depiction of a maximally ψ -epistemic model. Different probability distributions are plotted on hidden variable space λ . Supports of ρ_{ψ_2} and $\rho_{\psi_2^\perp}$ do not overlap due to orthogonality of $|\psi_2\rangle$ and $|\psi_2^\perp\rangle$. $R_{\psi_1\psi_2}$ is the overlap region of ρ_{ψ_2} and ρ_{ψ_1} . In this class of models every λ from the support of ρ_{ψ_1} belongs either to the support of ρ_{ψ_2} or $\rho_{\psi_2^\perp}$.

the cases) and it is known that without taking additional assumptions it is impossible to rule out an ontological model for qubits [20]. The results of [18] apply only to Hadamard states and the results from [17, 19] to PP-incompatible [21] ones.

In all these works only models close to maximally ψ -epistemic are refuted. The degree of “closeness” is treated differently in the papers. E. g. in [16] it is based on (6) and parameterized by Ω , which is between 0 and 1. The relation reads

$$p = \int_{R_{\psi_1\psi_2}} \rho_{\psi_1}(\lambda) d\lambda = \Omega |\langle \psi_2 | \psi_1 \rangle|^2 = \Omega \cos^2 \theta. \quad (7)$$

The maximally ψ -epistemic case corresponds to $\Omega = 1$ [15]. We call these models ψ_Ω -epistemic.

In [17–19] the overlap of probability distributions is measured differently. One can introduce another parameter k , again $0 < k < 1$ such that

$$\int \min\{\rho_{\psi_1}(\lambda), \rho_{\psi_2}(\lambda)\} d\lambda = k(1 - \sin \theta). \quad (8)$$

It’s relation with p is given by

$$p > k(1 - \sin \theta). \quad (9)$$

One can now get another definition of a maximally ψ -epistemic model by considering one with $k = 1$. We call such models ψ_k -epistemic.

VI. CRITICAL DETECTION EFFICIENCY FOR MAXIMALLY ψ -EPISTEMIC MODELS

By plugging either (7) or (9) and (5) into (3) and taking $\Omega = 1$ or $k = 1$, respectively, we obtain the critical value of detection efficiency as a function of θ only. The

results are plotted in Fig. 3 for ψ_Ω -epistemic and in Fig. 4 for ψ_k -epistemic models. The discontinuities in the graphs are due changes of the number of particle which is optimal for the given range of θ , see formula (5). It is easy to notice that the choice of θ which leads to the lowest detection efficiencies close to the point at which one has to change the number of qubits involved i.e. when $n^* = \frac{1}{\log_2(1 + \tan(\frac{\theta_{\min}}{2}))}$ is an integer.

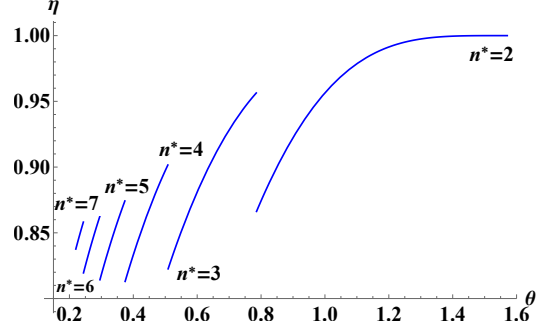


FIG. 3: Dependence of critical detection efficiency η as a function of θ measured in radians for different values of $n^* \in \{2, 3, \dots, 7\}$ for tests of maximally ψ_Ω -epistemic models.

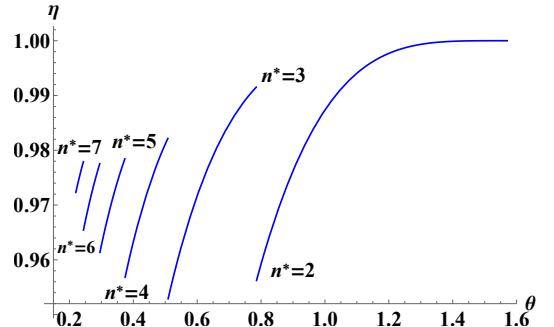


FIG. 4: Dependence of critical detection efficiency η as a function of θ measured in radians for different values of $n^* \in \{2, 3, \dots, 7\}$ for tests of maximally ψ_k -epistemic models.

We can also find the critical detection efficiency as a function of number of subsystems by taking θ ’s which correspond to integer values of n^* . These values are given in Table I. The minimal critical detection efficiency for testing maximally ψ_Ω -epistemic models is 81.3% and it is reached for an experiment with *four* subsystems while for testing maximally ψ_k -epistemic ones it is 95.3% and it requires *three* subsystems.

VII. NON-MAXIMALLY ψ -EPISTEMIC MODELS

Our analysis can be applied also for non-maximally ψ -epistemic models. It suffices to plug (7) or (9) and (5)

$\theta_{\min}(\text{in radians})$	n^*	η_{Ω}	η_k
0.785	2	0.866	0.956
0.509	3	0.822	0.953
0.374	4	0.813	0.957
0.295	5	0.814	0.961
0.244	6	0.819	0.965
0.207	7	0.826	0.969

TABLE I: Critical detection efficiencies for different number of subsystems (n^*). η_{Ω} and η_k correspond to maximally ψ_{Ω} -epistemic and ψ_k -epistemic models respectively.

into (3) and take any value of Ω or k between 0 and 1. This may be used for checking for which values of Ω and k there can be reasonable detection efficiency for a experimental test. Figs. 5 and 6 show the dependence of critical Ω and k as a function of the detection efficiency. Notice that around values of $\Omega \approx 0.7$ and $k \approx 0.9$, the required efficiency starts to be prohibitively high, as it reaches 97%, approximately the current state of the art.

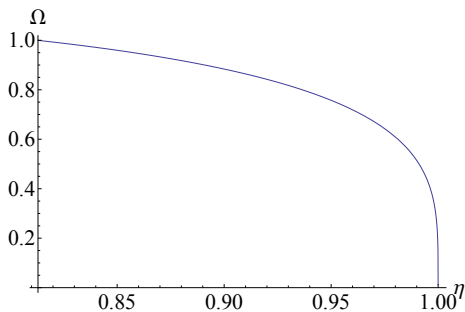


FIG. 5: Critical value of Ω as a function of detection efficiency η . For this plot optimal values of the parameters n^* and θ_{\min} have been used, ie. $n^* = 4$ and $\theta_{\min} = 0.374$.

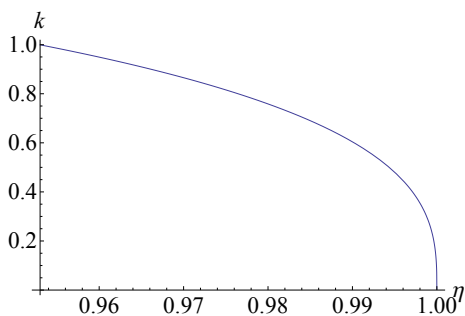


FIG. 6: Critical value of k as a function of detection efficiency η . For this plot optimal values of the parameters n^* and θ_{\min} have been used, ie. $n^* = 3$ and $\theta_{\min} = 0.509$.

VIII. CONCLUSIONS

We study the detection efficiency loophole in the context of PBR theorem. We point out that without additional assumptions the theorem only holds in the ideal case. For the non-deal case of inefficient detectors for maximally ψ -epistemic models, we obtain critical detection efficiency of 81.3%. If one uses a different definition of epistemicity, of the ψ_k type, the threshold increases to 95.3%. It is worth noting that this value is reached neither by the simplest case of *two* subsystems or in the limit of infinitely many but by an intermediate number (*three* in the first case and *four* in the second).

Our results show that the detection efficiency thresholds for quantum test based on the PBR gedanken-experiments are much higher than in the case of Bell inequalities. Not only it makes inconclusive possible experimental tests (see footnote [22]) which are done and analyzed without any other additional assumptions, but also it shows that if the strongest additional assumption concerning epistemicity is made the obtained critical value of detection efficiency is very high. Yet, it is almost within reach of the current state-of-the-art technology and we hope a loophole free refutation of maximally ψ -epistemic models would be performed soon.

Acknowledgments

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 - [22] Of course theorems cannot be tested. They are logical statements. But such tests have two-fold significance. They test whether the given prediction of quantum mechanics, on which the theorem is based, agrees with laboratory measurements, and whether the phenomena required in the theorem are experimentally observable (with current technology). If we have unconditional positive answers in both cases, then the thesis of the theorem may be thought of as a law of nature, or an expression of some law. Thus, in the end, we always test quantum mechanics, as a theory of nature.